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The rings of *n*-dimensional polytopes

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Abstract

Points of an orbit of a finite Coxeter group G, generated by n reflections starting from a single seed point, are considered as vertices of a polytope (G-polytope) centered at the origin of a real n-dimensional Euclidean space. A general efficient method is recalled for the geometric description of G-polytopes, their faces of all dimensions and their adjacencies. Products and symmetrized powers of G-polytopes are introduced and their decomposition into the sums of G-polytopes is described. Several invariants of G-polytopes are found, namely the analogs of Dynkin indices of degrees 2 and 4, anomaly numbers and congruence classes of the polytopes. The definitions apply to crystallographic and non-crystallographic Coxeter groups. Examples and applications are shown.

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1. Introduction

Finite groups generated by reflections in a real Euclidean space $\mathbb{R}^n$ of n dimensions, also called finite Coxeter groups, are split into two classes: crystallographic and non-crystallographic groups [1, 2]. The crystallographic groups are the Weyl groups of compact semisimple Lie groups. They are an efficient tool for uniform description of the semisimple Lie groups/algebras [3–5], and they have proven to be an indispensable tool in extensive computations with the representations of such Lie groups or Lie algebras (see for example [6] and references therein).

Underlying such applications are two facts: (i) most of the computation can be performed in integers by working with the weight systems of the representations involved in a problem and (ii) the weight system of a representation of a compact semisimple Lie group/Lie algebra consists of several Weyl group orbits of the weights, many of them occurring more than once. Practical importance of the orbits apparently emerged only in [7, 8], where truly large-scale computations were anticipated.

The crystallographic Coxeter groups are called Weyl groups and denoted by W. Any finite Coxeter group, crystallographic or not, is denoted by G. A difference between the two cases which is of practical importance to us is that lattices with W-symmetries are common crystallographic lattices, while lattices of non-crystallographic types are dense everywhere in $\mathbb{R}^n$.

Non-crystallographic finite Coxeter groups are of extensive use in modeling aperiodic point sets with long-range order ('quasicrystals') [9–11]. Outside traditional mathematics and mathematical physics, a new line of application of Coxeter group orbits can be found in [12]; see also the references therein.

Additional applications of Weyl group orbits are found in [13–17]. Both crystallographic and non-crystallographic Coxeter groups can be used for building families of orthogonal polynomials of many variables [18].

In recent years, another field of applications of W-orbits is emerging in harmonic analysis. Multidimensional Fourier-like transforms were introduced and are currently being explored in [19–22], where W-orbits are used to define families of special functions, called orbit functions [19], which serve as the kernels of the transforms. They differ from the traditional special functions [23]. The number of variables, on which the new functions depend, is equal to the rank of a compact semisimple Lie group that provides the Weyl group. Two properties of the transforms stand out: such special functions are orthogonal when integrated over a finite region F, and they are also orthogonal when summed up over lattice points $F_M \subset F$. The lattices can be of any density, their symmetries are prescribed by the Lie groups. Application of the non-crystallographic groups in Fourier analysis is at its very beginning [24].

In this paper, we have no compelling reason to distinguish crystallographic and non-crystallographic reflection groups of finite order. Hence, we consider all finite Coxeter groups although from the infinitely many finite Coxeter groups in two dimensions (symmetry groups of regular polygons), we usually consider only the lowest few.

An orbit $G(\lambda)$ of a Coxeter group G is the set of points in $\mathbb{R}^n$ generated by G from a single seed point $\lambda \in \mathbb{R}^n$. G-orbits are not common objects in the literature, nor is their multiplication, which can be viewed in parallel to the multiplication of G-invariant polynomials $P(\lambda; x)$ introduced in subsection 6.1 (for more about the polynomials see [18] and the references therein)[1]. Indeed, the set of exponents of all the monomials in $P(\lambda; x)$ is the set of points of the orbit $G(\lambda)$.

In this paper, we have adopted a point of view according to which the orbits $G(\lambda)$, being simpler than the polynomials $P(\lambda; x)$ or the weight systems of representations, are the primary objects of study.

The relation between the orbits of W and the weight systems of finite-dimensional irreducible representations of semisimple Lie groups/algebras over $\mathbb{C}$ can be understood as follows. The character of a particular representation involves summation over the weight system of the representation, i.e. over several W-orbits. As for which orbits appear in a particular representation, this is a well-known question about multiplicities of dominant weights. There is a laborious but rather fast computer algorithm for calculating the multiplicities. Extensive tables of multiplicities can be found in [25]; see also the references therein. Thus, one is justified in assuming that the relation between a representation and a particular W-orbit is known in all cases of interest.

Numerical characteristics, such as congruence classes, indices of various degrees and anomaly numbers, introduced here for W-orbits, mirror similar properties of weight systems from representation theory, which are often used in applications (for example [13–15, 26, 27]).

[1] Polynomials in 6.1 are the simplest W-invariant ones. We are not concerned about any other of their properties.

In this paper, we introduce operations on W-orbits that are well known for weight systems of representations: (i) the product of W-orbits (of the same group) and its decomposition into the sum of W-orbits; (ii) the decomposition of the kth power of a W-orbit symmetrized by the group of permutations of k elements. New is the introduction of such operations for the orbits of non-crystallographic Coxeter groups. We intend to describe reductions of G-orbits to orbits of a subgroup $G' \subset G$ in a separate paper [28]. Again, the involvement of non-crystallographic groups makes the reduction problem rather unusual. Corresponding applications deserve to be explored.

The decomposition of products of orbits of Coxeter groups, as introduced here, is the core of other decomposition problems in mathematics, such as the decomposition of direct products of representations of semisimple Lie groups, the decomposition of products of certain special functions [19] and the decomposition of products of G-invariant polynomials of several variables [18]. The last two problems are completely solved in terms of orbit decompositions. The first problem requires that the multiplicities of dominant weights in weight systems of representations [25] be known.

We view the G-orbits from a perspective uncommon in the literature. Namely, the points of a G-orbit are taken to be vertices of an n-dimensional G-invariant polytope centered at origin, n being the number of elementary reflections generating G (at the same time it is the rank of the corresponding semisimple Lie group). The multiplication of two such polytopes/orbits, say P_1 and P_2, is the set of points/vertices obtained by adding to every point of P_1 every point of P_2. The resulting set of points is again G-invariant and thus it is a union (we say 'sum') of several G-orbits (we say 'G-polytopes'). Thus, we have a ring of G-polytopes with positive integer coefficients. We recall and illustrate a general method of description of n-dimensional reflection-generated polytopes [29, 30].

The core of our geometric interpretation of orbits as polytopes is in the paragraph following equation (12). A product of orbits is a union of concentric orbits. Geometrically, this can be seen as an 'onion'-like layered structure of orbits of different radii. Unlike in representation theory, where orbit points are always points of the corresponding weight lattices, in our case the seed point of an orbit can be anywhere in $\mathbb{R}^n$. In particular, a suitable choice of the seed points of the orbits, which are being multiplied, can bring some of the layers of the 'onion' structure as close or as far apart as desired. Two examples are given in the last section (see (35) and (36)).

2. Reflections generating finite Coxeter groups

Let α and x be vectors in $\mathbb{R}^n$. We denote by r_α the reflection in the $(n-1)$-dimensional 'mirror' orthogonal to α and passing through the origin. For any $x \in \mathbb{R}^n$, we have

$$r_\alpha x = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha. \tag{1}$$

Here $\langle a, b\rangle$ denotes scalar product in $\mathbb{R}^n$. In particular, we have $r_\alpha 0 = 0$ and $r_\alpha \alpha = -\alpha$ so that $r_\alpha^2 = 1$.

A Coxeter group G is by definition generated by several reflections in mirrors that have the origin as their common point. Various Coxeter groups are thus specified by the set $\Pi(\alpha)$ of vectors α, orthogonal to the mirrors and called the simple roots of G. Consequently, G is given once the relative angles between elements of $\Pi(\alpha)$ are given.

A standard presentation of G, generated by n reflections, amounts to the following relations:

$$r_k^2 = 1, \qquad (r_i r_j)^{m_{ij}} = 1, \qquad k, i, j \in \{1, \dots, n\},$$

where we have simplified the notation by setting $r_{\alpha_k} = r_k$, and where m_{ij} are the lowest possible positive integers. The matrix (m_{ij}) specifies the group. The angles between the mirrors of reflections r_i and r_j are determined from the values of the exponents m_{ij}. Indeed, for $m_{ij} = p$, the angle is π/p, while the angle between α_i and α_j is $\pi - \pi/p$.

The classification of finite reflection (Coxeter) groups was accomplished in the first half of the 20th century.

2.1. $n = 1$

There is just one group of order 2. Its two elements are 1 and r. We denote this group by A_1. Acting on a point a of the real line, the group A_1 generates its orbit of two points, a and $ra = -a$, except if $a = 0$. Then the orbit consists of just one point, namely the origin.

2.2. $n = 2$

There are infinitely many Coxeter groups in $\mathbb{R}^2$, one for each $m_{12} = 2, 3, 4, \ldots$. Their orders are $2m_{12}$. In physics literature, these are the dihedral groups.

Note that for $m_{12} = 2$, the group is a product of two groups from $n = 1$. The reflection mirrors are orthogonal.

Our notation for the lowest five groups, generated by two reflections, and their orders, is as follows:

$$
\begin{array}{lll}
m_{12} = 2: & A_1 \times A_1, & 4 \\
m_{12} = 3: & A_2, & 6 \\
m_{12} = 4: & C_2, & 8 \\
m_{12} = 5: & H_2, & 10 \\
m_{12} = 6: & G_2, & 12.
\end{array}
$$

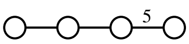
2.3. General case: Coxeter and Dynkin diagrams

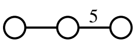
A convenient general way to provide a specific set $\Pi(\alpha)$ is to draw a graph where vertices are traditionally shown as small circles, one for each $\alpha \in \Pi$, and where edges indicate absence of orthogonality between two vertices linked by an edge.

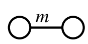
A diagram consisting of several disconnected components means that the group is a product of several pairwise commuting subgroups. Thus it is often sufficient to consider only the groups with connected diagrams.

In this paper, a Coxeter diagram is a graph providing only relative angles between simple roots while ignoring their lengths. This is done by writing m_{ij} over the edges of the diagram. By convention, the most frequently occurring value, $m_{ij} = 3$, is not shown in the diagrams. When $m_{ij} = 2$, the edge is not drawn, i.e. the nodes numbered i and j are not directly connected.

Consider the examples of Coxeter diagrams of all finite non-crystallographic Coxeter groups with connected diagrams. Note that we simply write H_2 when $m = 5$.

H_4 

H_3 

$H_2(m)$ 

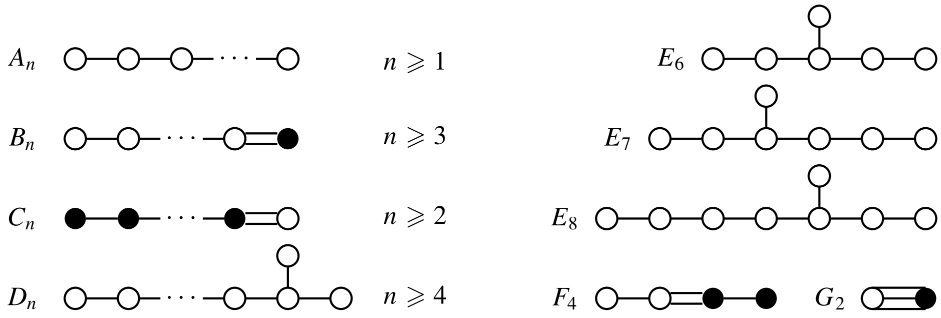
$m = 5, 7, 8, 9, \ldots.$

A Dynkin diagram is a graph providing, in addition to the relative angles, the relative lengths of the vectors from $\Pi(\alpha)$. Dynkin diagrams are used for the crystallographic Coxeter groups, frequently called the Weyl groups. There are four infinite series of classical groups

Table 1. Orders of the finite Coxeter groups.

A_n ($n \geqslant 1$)	B_n ($n \geqslant 3$)	C_n ($n \geqslant 2$)	D_n ($n \geqslant 4$)	E_6	E_7
$(n+1)!$	$2^n n!$	$2^n n!$	$2^{n-1} n!$	$2^7 3^4 5$	$2^{10} 3^4 5 7$
E_8	F_4	G_2	$H_2(m)$	H_3	H_4
$2^{14} 3^5 5^2 7$	$2^7 3^2$	12	$2m$	120	120^2

and five isolated cases of exceptional simple Lie groups. Here is a complete list of Dynkin diagrams of such groups (with connected diagrams):



The names of the groups, as is traditional in Lie theory, are shown on the left of each diagram. Open (black) circles indicate longer (shorter) roots. The ratio of their square lengths is $\langle\alpha_l, \alpha_l\rangle$:$\langle\alpha_s, \alpha_s\rangle = 2$:1 in all cases except for G_2 where the ratio is 3:1. Moreover, we adopt the usual convention that $\langle\alpha_l, \alpha_l\rangle = 2$. A single, double and triple line indicates respectively the angle $2\pi/3$, $3\pi/4$ and $5\pi/6$ between the roots, or equivalently, the angles $\pi/3$, $\pi/4$ and $\pi/6$ between the reflection mirrors. The absence of a direct link between two nodes implies that the corresponding simple roots, as well as the mirrors, are orthogonal. Note that the relative angles of the mirrors of B_n and C_n coincide. Hence their W-groups are isomorphic. Their simple roots differ by length.

We adopt the Dynkin numbering of nodes. The numbering proceeds from left to right $1, 2, \ldots$. In the case of D_n and E_6, E_7, E_8, the node above the main line has the highest number, respectively n, 6, 7, 8.

Orders of the finite Coxeter groups are provided in table 1 for groups with connected diagrams. When a diagram has several disconnected components, the order is the product of orders corresponding to each subdiagram.

3. Root and weight lattices

Information essentially equivalent to that provided by the Coxeter and Dynkin diagrams is also given in terms of $n \times n$ matrices C, called the Cartan matrices. Relative angles and lengths of simple roots can be used to form the Cartan matrix for each group. Its matrix elements are calculated as

$$C = (C_{jk}) = \left(\frac{2\langle\alpha_j, \alpha_k\rangle}{\langle\alpha_k, \alpha_k\rangle}\right), \qquad j, k \in \{1, 2, \ldots, n\}. \tag{2}$$

Cartan matrices and their inverses are given in many places, e.g., [1, 25].

The Cartan matrices can be defined for any finite Coxeter group by using formula (2). For non-crystallographic groups the matrices are

$$C(H_2) = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \qquad C(H_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix},$$

$$C(H_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix},$$

where τ is the larger of the solutions of the algebraic equation $x^2 = x + 1$, i.e. $\tau = \frac{1}{2}(1+\sqrt{5})$.

In addition to the basis of simple roots (α-basis), it is useful to introduce the basis of fundamental weights (ω-basis). Subsequently, most of our computations will be performed in the ω-basis:

$$\alpha = C\omega, \qquad \omega = C^{-1}\alpha.$$

Note the important relation

$$\langle \alpha_k, \omega_j \rangle = \delta_{jk} \frac{\langle \alpha_k, \alpha_k \rangle}{2}, \qquad j, k \in \{1, 2, \ldots, n\}. \tag{3}$$

Illustrations showing the α- and ω-bases of A_2, C_2 and G_2 are given in figure 1 of [29].

The root lattice Q and the weight lattice P of G are formed by all integer linear combinations of simple roots, respectively fundamental weights, of G,

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n, \qquad P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n. \tag{4}$$

Here $\mathbb{Z}$ stands for any integer. For the groups that have simple roots of two different lengths, one may define the root lattice of n linearly independent short roots, which cannot all be simple. In general, $Q \subseteq P$, with $Q = P$ only for E_8, F_4 and G_2.

If G is one of the non-crystallographic Coxeter groups, the lattices Q and P are dense everywhere.

Since α- and ω-bases are not orthogonal and not normalized, it is sometimes useful to work with orthonormal bases. For crystallographic groups, they are found in many places, for example [3, 25]. For non-crystallographic groups, H_2, H_3 and H_4; see [11, 9].

4. The orbits of Coxeter groups

4.1. Computing points of an orbit

Given the reflections r_α, $\alpha \in \Pi(\alpha)$, of a Coxeter group G, and a seed point $\lambda \in \mathbb{R}^n$, the points of the orbit $G(\lambda)$ are given by the set of distinct points generated by repeated application of the reflections r_α to λ. All points of an orbit are equidistant from the origin. The radius of an orbit is the distance of (any) point of the orbit from the origin.

There are practically important considerations which make it almost imperative that the computation of the points of any orbit of G be carried out in the ω-basis, as follows:

- Every orbit contains precisely one point with nonnegative coordinates in the ω-basis. We specify the orbit by that point, calling it the dominant point of the orbit.
- Given a dominant point λ of the group G in the ω-basis, one readily finds the size of the orbit $G(\lambda)$, i.e. the number of points in the orbit, using the order $|G|$ of the Coxeter group and the order of the stabilizer of λ in G:

$$|G(\lambda)| = \frac{|G|}{|\mathrm{Stab}_G(\lambda)|}. \tag{5}$$

Here $\mathrm{Stab}_G(\lambda)$ is a Coxeter subgroup of G. To find it, one needs to attach the ω-coordinates of λ to the corresponding nodes of the diagram of G. The subdiagram carrying the coordinates 0 is the diagram of $\mathrm{Stab}_G(\lambda)$.

- Due to (3), the reflections (1) are particularly simple when applied to ω's:

$$r_k\omega_j = \omega_j - \frac{2\langle\alpha_k, \omega_j\rangle}{\langle\alpha_k, \alpha_k\rangle}\alpha_k = \omega_j - \delta_{jk}\alpha_k. \tag{6}$$

- Starting from the dominant point of an orbit, it suffices to apply, during the computation of the orbit points, only reflections corresponding to positive coordinates of any given weight. All points of the orbit are found in this way.

4.2. Orbits of A_2, C_2, G_2 and H_2

We give some examples of orbits. Let $a, b > 0$:

$$\begin{aligned} A_2: \quad G((a,0)) &= \{(a,0), (-a,a), (0,-a)\},\\ G((0,b)) &= \{(0,b), (b,-b), (-b,0)\},\\ G((a,b)) &= \{(a,b), (-a,a+b), (b,-a-b), (a+b,-b),\\ &\qquad (-a-b,a), (-b,-a)\}. \end{aligned}$$

In particular, the orbit $G((1,1)) = \{(1,1), (-1,2), (1,-2), (2,-1), (-2,1), (-1,-1)\}$ consists of the vertices of a regular hexagon of radius $\sqrt{2}$. It is the root system of A_2:

$$\begin{aligned} C_2: \quad G((a,0)) &= \{\pm(a,0), \pm(-a,a)\},\\ G((0,b)) &= \{\pm(0,b), \pm(2b,-b)\},\\ G((a,b)) &= \{\pm(a,b), \pm(-a,a+b), \pm(a+2b,-a-b),\\ &\qquad \pm(-a-2b,b)\}. \end{aligned}$$

In particular, the orbits $G((2,0))$ and $G((0,1))$ of radii $\sqrt{2}$ and 1 are respectively the vertices and midpoints of the sides of a square. Together the two orbits form the root system of C_2:

$$\begin{aligned} G_2: \quad G((a,0)) &= \{\pm(a,0), \pm(-a,3a), \pm(2a,-3a)\},\\ G((0,b)) &= \{\pm(0,b), \pm(b,-b), \pm(-b,2b)\},\\ G((a,b)) &= \{\pm(a,b), \pm(-a,3a+b), \pm(2a+b,-3a-b),\\ &\qquad \pm(-2a-b,3a+2b), \pm(a+b,-3a-2b),\\ &\qquad \pm(-a-b,b)\}. \end{aligned}$$

In particular, the orbits $G((1,0))$ and $G((0,1))$ are the vertices of regular hexagons of radii $\sqrt{2}$ and $2/\sqrt{3}$, rotated relatively by 30°, i.e. they form a hexagonal star. Together the two orbits form the root system of G_2. The points of $G((a, a\sqrt{3}/\sqrt{2}))$, $a > 0$, are the vertices of a regular dodecahedron of radius $\sqrt{2}a$:

$$\begin{aligned} H_2: \quad G((a,0)) &= \{(a,0), (-a,a\tau), (a\tau,-a\tau), (-a\tau,a), (0,-a)\},\\ G((0,b)) &= \{(0,b), (b\tau,-b), (-b\tau,b\tau), (b,-b\tau), (-b,0)\},\\ G((a,b)) &= \{(a,b), (-a,b+a\tau), (a\tau+b\tau,-b-a\tau),\\ &\qquad (-a\tau-b\tau,a+b\tau), (b,-a-b\tau), (a+b\tau,-b),\\ &\qquad (-a-b\tau,a\tau+b\tau), (b+a\tau,-a\tau-b\tau),\\ &\qquad (-b-a\tau,a), (-b,-a)\}. \end{aligned}$$

In particular, the orbits $G((a,0))$ and $G((0,b))$ are the vertices of regular pentagons of radii $a\sqrt{2}$ and $b\sqrt{2}$, rotated relatively by 36°. The orbit $G((a,a))$ forms a regular decahedron. The orbit $G((\tau,\tau))$ consists of the roots of H_2.

An orbit of A_2 or H_2 contains, with every point (p, q) also the point $(-q, -p)$. Note that in the examples of this subsection the constants a and b do not need to be integers. All one requires is that they are positive. Effects of special choices of these constants are exemplified in (35) and (36).

4.3. Orbits of A_3, B_3, C_3 and H_3

We give some examples of orbits. Let $a, b > 0$:

$$\begin{aligned}
A_3: \quad & G((a,0,0)) = \{(a,0,0), (-a,a,0), (0,-a,a), (0,0,-a)\},\\
& G((0,b,0)) = \{\pm(0,b,0), \pm(b,-b,b), \pm(-b,0,b)\},\\
& G((1,1,0)) = \{(1,1,0), (-1,2,0), (2,-1,1), (1,-2,2), (-2,1,1),\\
& \qquad (2,0,-1), (-1,-1,2), (1,0,-2), (-2,2,-1),\\
& \qquad (-1,1,-2), (0,-2,1), (0,-1,-1)\},\\
B_3: \quad & G((a,0,0)) = \{\pm(a,0,0), \pm(-a,a,0), \pm(0,-a,2a)\},\\
& G((0,b,0)) = \{\pm(0,b,0), \pm(b,-b,2b), \pm(-b,0,2b), \pm(b,b,-2b),\\
& \qquad \pm(-b,2b,-2b), \pm(2b,-b,0)\},\\
& G((0,0,c)) = \{\pm(0,0,c), \pm(0,c,-c), \pm(c,-c,c), \pm(c,0,-c)\}.\\
C_3: \quad & G((a,0,0)) = \{\pm(a,0,0), \pm(-a,a,0), \pm(0,-a,a)\},\\
& G((0,b,0)) = \{\pm(0,b,0), \pm(b,-b,b), \pm(-b,0,b), \pm(b,b,-b),\\
& \qquad \pm(-b,2b,-b), \pm(2b,-b,0)\},\\
& G((0,0,c)) = \{\pm(0,0,c), \pm(0,2c,-c), \pm(2c,-2c,c), \pm(2c,0,-c)\}.\\
H_3: \quad & G((a,0,0)) = \{\pm(a,0,0), \pm(-a,a,0), \pm(0,-a,a\tau), \pm(0,a\tau,-a\tau),\\
& \qquad \pm(a\tau,-a\tau,a), \pm(-a\tau,0,a)\}.\\
& G((0,0,c)) = \{\pm(0,0,c), \pm(0,\tau c,-c), \pm(\tau c,-\tau c,\tau^2 c), \pm(-\tau c,0,\tau^2 c),\\
& \qquad \pm(a\tau,-a\tau,a), \pm(-a\tau,0,a)\}.
\end{aligned}$$

5. Orbits as polytopes

In this section, we recall an efficient method [30] of description for reflection-generated polytopes in any dimension.

The idea of the method consists in the following. Suppose we have an orbit $G(\lambda)$. Consider its points as vertices (faces of dimension 0) of the polytope also denoted $G(\lambda)$ in $\mathbb{R}^n$. Then for any face f of dimension $0 \leqslant d \leqslant n-1$, we identify its stabilizer $\mathrm{Stab}_{G(\lambda)}(f)$ in G, which is a product of two Coxeter subgroups of G:

$$\mathrm{Stab}_{G(\lambda)}(f) = G_1(f) \times G_2(D)$$

where $G_1(f)$ is the symmetry group of the face, and $G_2(D)$ stabilizes f pointwise, i.e., does not move it at all.

Our method consists in recursive decorations of the diagram of G, providing at each stage the subdiagrams of $G_1(f) = G(\star)$ and $G_2(D) = G(\circ)$ for faces of one type. The decoration of the nodes of the diagram indicates to which $G(\star)$ or $G(\circ)$ subgroups of the stabilizer the corresponding reflections belong. For further details, see [30]. A much wider application of this method is described in [29, 31, 32], including its exploitation in non-Euclidean spaces.

Table 2. Number of faces of $2D$ polytopes with Coxeter group symmetry. The first three rows specify representatives of G-orbits of $2D$ polytopes. A black (open) dot in the second column stands for a positive (zero) coordinate in the ω-basis of the dominant point representing the orbit of vertices. The number of vertices is listed in the subsequent five columns. Rows 4 and 5 refer to the edges of the polytopes. A star in the second column indicates the reflection generating the symmetry group of the edge. The number of edges is shown for each group in subsequent columns. Check marks in one of the last three columns indicate the faces which belong to the polytope described in that column.

		A_2	C_2	G_2	H_2	$H_2(7)$	1	2	3
1	●●	6	8	12	10	14	✓		
2	●○	3	4	6	5	7		✓	
3	○●	3	4	6	5	7			✓
4	⋆●	3	4	6	5	7	✓	✓	
5	●⋆	3	4	6	5	7	✓		✓

Table 3. The $3D$ polytopes generated by a Coxeter group from a single seed point, and the number of their faces of dimension 0, 1 and 2. Decorated diagrams, rows 1 to 7, specify the polytopes. The dimension of a face equals the number of stars in the diagram. See 5.1 for additional explanations.

	Diagram	A_3	B_3	C_3	H_3	1	2	3	4	5	6	7
1	● ● ●	24	48	48	120	✓						
2	● ● ○	12	24	24	60		✓					
3	● ○ ●	12	24	24	60			✓				
4	○ ● ●	12	24	24	60				✓			
5	● ○ ○	4	6	6	12					✓		
6	○ ● ○	6	12	12	30						✓	
7	○ ○ ●	4	8	8	20							✓
8	⋆ ● ●	12	24	24	60	✓		✓				
9	● ⋆ ●	12	24	24	60	✓	✓		✓		✓	
10	● ● ⋆	12	24	24	60	✓		✓				
11	⋆ ● ○	6	12	12	30		✓			✓		
12	○ ● ⋆	6	12	12	30				✓			✓
13	⋆ ⋆ ●	4	8	8	20	✓	✓	✓	✓	✓	✓	
14	⋆ ● ⋆	6	12	12	30	✓		✓				
15	● ⋆ ⋆	4	6	6	12	✓	✓	✓	✓		✓	✓

We start with an extreme decoration of the diagram. It is equivalent to stating which coordinates of the dominant weight are positive relative to the ω-basis. The nodes are drawn as either open or black circles, i.e. zero or positive coordinates respectively.

Every possible extreme decoration fixes a polytope. There are only two rules for recursive decoration of the diagrams, starting from one of the extreme ones: (i) A single black circle is replaced by a star; (ii) open circles, that become adjacent to a star by diagram connectivity, are changed to black ones.

Tables 2 and 3 show the results of the application of the decoration rules for polytopes in $2D$ and $3D$ for all groups with connected diagrams. All polytopes for A_4, B_4, C_4, D_4 and H_4 are described in tables 3 and 4 of [30].

5.1. Explanation of the tables

A description of table 2 is given in its caption.

Consider table 3. The second column contains short-hand notation for several diagrams at once. We call them decorated diagrams. No links between nodes of a diagram are drawn because they would need to be different for each group in subsequent columns. The nodes do not reveal the relative lengths of roots, their decoration indicates to which of the pertinent subgroup of the stabilizer of G such a reflection belongs. Thus the diagrams of the second column of the table apply to A_3, B_3, C_3 and H_3 at the same time.

Each line of the table describes one of G-orbits of identical faces. The dimension of the face equals to the number of stars in its decorated diagram. Numerical entries in a row give the number of faces for polytopes of symmetry groups A_3, B_3, C_3 and H_3, shown in the header of the columns. The top seven rows show the starting decorations fixing the polytopes, and also the number of 0-faces (vertices) of the polytopes of each group. The check marks in one of the last seven columns indicate the faces belonging to the same polytope.

Example 1. As an example of how to decipher properties of polytope faces, consider rows number 5 and 2. The diagram in row 5 conveys the fact that $\lambda = a\omega_1$ with $a > 0$. The exact value of a affects only the size of the polytope, not its shape. The stabilizer of λ is given by the subdiagram of open circles, i.e. r_2 and r_3 generate its stabilizer. For A_3 the subdiagram is of type A_2, while for B_3 and C_3 it is of type C_2, and for H_3 it is of type H_2. Hence in row 5 the entries give the number of vertices as $24/6$, $48/8$, $48/8$, $120/10$ respectively.

The check mark in column 5 and row 5 indicates that faces belonging to our polytope are indicated by other check marks in column 5, namely in rows 11 and 13. The diagram of row 11 has just one star, hence the face is one-dimensional (an edge). Its stabilizer (the subdiagram of stars and open circles) is of type $A_1 \times A_1$ for all four cases. Hence the number of edges is $24/4$ for A_3, $48/4$ for B_3 and C_3, and $120/4$ for H_3. The only type of $2D$ face is given in row 13. The symmetry group of the face is generated by r_1 and r_2. It is of type A_2 for all four cases. Thus there are $24/6$ faces in A_3, $48/6$ in B_3 and C_3, and $120/6$ in H_3 polytope.

Similarly, row 2 indicates that $\lambda = a\omega_1 + b\omega_2, a, b > 0$. It is stabilized by the group generated by r_3, which is of type A_1 for all four cases. Hence the number of vertices equals half of the order of the corresponding Coxeter group. There are two orbits of edges given in rows 9 and 11, while the two orbits of $2D$ faces are given by the check marks in rows 13 and 15.

Example 2. The $2D$ faces can actually be constructed knowing their symmetry and the seed point, say $(a, 0, 0)$. The diagram of the $2D$ face is $\star \star \bullet$, meaning that the symmetry group of the face is generated by r_1 and r_2. Moreover, it is of the same type (A_2) for all four groups. Then there are just three distinct vertices of the $2D$ face:

$$(a, 0, 0), \qquad r_1(a, 0, 0), \qquad r_2r_1(a, 0, 0).$$

The $2D$ face is formed from the seed point $(a, 0, 0)$ by application of reflections r_1 and r_2.

The vertices of the $2D$ face are different triangles for each group, because they are given in their respective ω-basis:

$$\begin{aligned}
A_3 &: \quad (a, 0, 0), (-a, a, 0), (0, -a, a),\\
B_3 &: \quad (a, 0, 0), (-a, a, 0), (0, -a, 2a),\\
C_3 &: \quad (a, 0, 0), (-a, a, 0), (0, -a, a),\\
H_3 &: \quad (a, 0, 0), (-a, a, 0), (0, -a, a\tau).
\end{aligned}$$

Example 3. Let us consider row 2 in further detail. The starting point is $\lambda = a\omega_1 + b\omega_2$, where $a, b > 0$. There are two orbits of edges given by their endpoints:

$$((a, b, 0), r_1(a, b, 0)), \qquad ((a, b, 0), r_2(a, b, 0)),$$

and two orbits of $2D$ faces. Consider just the H_3 case. The $2D$ face of row 13 has the symmetry group generated by r_1, r_2 (A_2 type). It is a hexagon:

$$(a, b, 0), \qquad (-a, a+b, 0), \qquad (a+b, -b, \tau b), \qquad (b, -a-b, \tau(a+b)),$$
$$(-a-b, a, \tau b), \qquad (-b, -a, \tau(a+b)).$$

The $2D$ face of row 15 has its symmetry group generated by r_2, r_3 (H_2 type). It is a pentagon:

$$(a, b, 0), \quad (a+b, -b, \tau b), \quad (a+b, \tau b, -\tau b),$$
$$(a+\tau^2 b, -\tau b, b), \quad (a+\tau^2 b, 0, -b).$$

In particular, when $a = b$, the pentagon and the hexagon are both regular. The polytope is then the familiar fullerene or 'soccer ball'.

Further questions about the structure of polytopes can be answered within our formalism: How many $2D$ faces meet in a vertex? Which $2D$ faces meet in an edge? The higher the dimension, the more questions like these can be asked and answered. For more information on such questions and others (e.g. dual pairs of polytopes), we refer to [30].

6. Decomposition of products of polytopes

6.1. Multiplication of G-invariant polynomials

The product of G-polytopes together with its decomposition, as defined in section 6.2, can be simply motivated by its correspondence to the product of more familiar objects than orbits, namely G-invariant polynomials, say $P(\lambda; x)$ and $P(\mu; x)$. Here λ and μ are dominant points of their orbits and x stands for n auxiliary independent variables $x_1, x_2, \ldots, x_n$ whose nature is of no concern to us here. They can be thought of as, for example, complex or real variables. We introduce them in order to make sense of the definitions below.

Denote by $\lambda^{(i)} \in G(\lambda)$ the points of the orbit $G(\lambda)$, and by $\mu^{(k)} \in G(\mu)$, where

$$\lambda^{(i)} = \sum_{p=1}^{n} a_p^{(i)} \omega_p, \quad \mu^{(k)} = \sum_{q=1}^{n} b_q^{(k)} \omega_q, \qquad 1 \leqslant i \leqslant |G(\lambda)|, \quad 1 \leqslant k \leqslant |G(\mu)|. \tag{7}$$

Here $|G(\lambda)|$ and $|G(\mu)|$ denote the number of points in their orbits. Then we can introduce the polynomials:

$$P(\lambda; x) = \sum_{\lambda^{(i)} \in G(\lambda)} x^{\lambda^{(i)}} := \sum_{i=1}^{|G(\lambda)|} x_1^{a_1^{(i)}} x_2^{a_2^{(i)}} \cdots x_n^{a_n^{(i)}}, \tag{8}$$

$$P(\mu; x) = \sum_{\mu^{(k)} \in G(\mu)} x^{\mu^{(k)}} := \sum_{k=1}^{|G(\mu)|} x_1^{b_1^{(k)}} x_2^{b_2^{(k)}} \cdots x_n^{b_n^{(k)}}, \tag{9}$$

and their product,

$$P(\lambda; x) \otimes P(\mu; x) = \sum_{i=1}^{|G(\lambda)|} \sum_{k=1}^{|G(\mu)|} x_1^{a_1^{(i)}+b_1^{(k)}} x_2^{a_2^{(i)}+b_2^{(k)}} \cdots x_n^{a_n^{(i)}+b_n^{(k)}}. \tag{10}$$

The latter consists of the sum of $|G(\lambda)||G(\mu)|$ monomials which can be decomposed into the sum of polynomials defined by one G-orbit each.

Finally, consider an example: let G be the group A_2, and $\lambda = (1, 0)$ and $\mu = (0, 1)$. Therefore $P((1, 0); x) = x_1 + x_1^{-1}x_2 + x_2^{-1}$ and $P((0, 1); x) = x_2 + x_1x_2^{-1} + x_1^{-1}$. Their products decompose as follows:

$$\begin{aligned} P((1,0);x) \otimes P((0,1);x) &= \left\{x_1x_2 + x_1^2x_2^{-1} + x_1^{-1}x_2^2 + x_1^{-2}x_2 + x_1x_2^{-2} + x_1^{-1}x_2^{-1}\right\} + 3 \\ &= P((1,1);x) + 3P((0,0);x), \\ P((1,0);x) \otimes P((1,0);x) &= \left\{x_1^2 + x_1^{-2}x_2^2 + x_2^{-2}\right\} + 2\left\{x_2 + x_1x_2^{-1} + x_1^{-1}\right\} \\ &= P((2,0);x) + 2P((0,1);x). \end{aligned}$$

6.2. *Products of G-orbits*

Suppose we are given two orbits, say $G(\lambda)$ and $G(\mu)$, of the same Coxeter group G. Let $\lambda^{(i)}$ and $\mu^{(k)}$ be the points of $G(\lambda)$ and $G(\mu)$ respectively, numbered in some way. We define the product of two orbits as

$$G(\lambda) \otimes G(\mu) := \bigcup_{\lambda^{(i)} \in G(\lambda), \mu^{(k)} \in G(\mu)} (\lambda^{(i)} + \mu^{(k)}). \tag{11}$$

The left side is obviously G-invariant, therefore the right side is also G-invariant. Hence it can be decomposed into a union of several G-orbits. The highest and the lowest components of such a decomposition are easily obtained:

$$G(\lambda) \otimes G(\mu) = G(\lambda + \mu) \cup \cdots \cup G(\lambda + \overline{\mu}). \tag{12}$$

Here, $\lambda + \mu$ is the sum of the dominant points of the orbits $G(\lambda)$ and $G(\mu)$. The symbol $\overline{\mu}$ stands for the unique lowest point of $G(\mu)$ (all coordinates are non-positive in the ω-basis). Frequently, it happens that $\lambda + \overline{\mu}$ is not a dominant point, i.e. the highest point in its orbit, but it still identifies the orbit uniquely. Note also that $\lambda + \overline{\mu}$ and $\mu + \overline{\lambda}$ always belong to the same G-orbit. The lowest component often appears more than once in the decomposition.

For a geometric interpretation of (12), recall that all orbits in (12) are concentric, having the origin as their common center, and that points of one orbit are equidistant from the origin. In physics, the product on the left side of (12) can be thought of as a certain 'interaction' between two orbit layers, resulting on the right side in an 'onion'-like structure of several concentric orbit layers.

To simplify the notation in the following examples, we write just λ instead of $G(\lambda)$, so that $\lambda \otimes \mu$ means $G(\lambda) \otimes G(\mu)$.

6.3. *Two-dimensional examples*

$$\begin{aligned} A_2: \quad & (1,0) \otimes (0,1) = (1,1) \cup 3(0,0), \\ & (1,0) \otimes (1,1) = (2,1) \cup 2(1,0) \cup 2(0,2), \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(1,1) \cup 2(3,0) \cup 2(0,3) \cup 6(0,0). \\ C_2: \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(1,0), \\ & (1,0) \otimes (1,1) = (2,1) \cup 2(2,0) \cup 2(0,2) \cup 2(0,1), \\ & (1,1) \otimes (1,1) = (2,2) \cup 2(2,1) \cup 2(4,0) \cup 2(2,0) \cup 2(0,3) \\ & \qquad\qquad\qquad \cup 2(0,1) \cup 8(0,0). \\ G_2: \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(0,2) \cup 2(0,1), \\ & (1,0) \otimes (1,1) = (2,1) \cup (1,2) \cup (1,1) \cup 2(0,4) \cup 2(0,2) \cup 2(0,1), \end{aligned}$$

$$
\begin{aligned}
&(1,1)\otimes(1,1) = (2,2)\cup 2(1,1)\cup 2(1,3)\cup 2(3,0)\cup 2(2,0)\\
&\qquad\qquad\qquad\quad \cup 2(1,0)\cup 2(0,5)\cup 2(0,4)\cup 2(0,1)\cup 12(0,0).
\end{aligned}
$$

$$
\begin{aligned}
H_2:\quad &(1,0)\otimes(0,1) = (1,1)\cup(\tau-1,\tau-1)\cup 5(0,0),\\
&(1,0)\otimes(1,1) = (2,1)\cup(\tau,\tau-1)\cup(\tau-1,1)\cup 2(1,0)\cup 2(0,\tau+1),\\
&(1,1)\otimes(1,1) = (2,2)\cup 2(\tau,\tau)\cup 2(\tau-1,\tau-1)\cup 2(2+\tau,0)\\
&\qquad\qquad\qquad\quad \cup 2(2\tau-1,0)\cup 2(0,2+\tau)\cup 2(0,2\tau-1)\cup 10(0,0).
\end{aligned}
$$

6.4. Three-dimensional examples

$$
\begin{aligned}
A_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 4(0,0,0),\\
&(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 3(2,0,0)\cup 4(0,1,0)\cup 3(0,0,2),\\
&(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 3(2,0,0)\cup 2(0,1,0).\\
B_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 3(0,0,1),\\
&(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,0,1)\cup 3(1,0,1)\cup 2(0,1,1)\\
&\qquad\qquad\qquad\qquad\quad \cup 3(0,0,3)\cup 6(0,0,1),\\
&(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,0,1)\cup 2(1,0,1)\cup 2(0,1,1).\\
C_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup 2(0,1,0),\\
&(1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(2,0,0)\\
&\qquad\qquad\qquad\qquad\quad \cup 4(0,2,0)\cup 4(0,1,0)\cup 3(0,0,2),\\
&(1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(0,1,0).\\
H_3:\quad &(1,0,0)\otimes(0,0,1) = (1,0,1)\cup(0,\tau-1,\tau-1)\cup 5(\tau,0,0)\\
&\qquad\qquad\qquad\qquad\quad \cup 3(0,0,\tau-1).
\end{aligned}
$$

6.5. Decomposition of products of E_8 orbits

We say that an orbit is fundamental if its dominant weight in the ω-basis has precisely one coordinate equal to 1 and all others are zero. Thus E_8 has eight fundamental orbits. Their sizes range from 240 to over 17 000.

All 36 different products of fundamental orbits of E_8 were decomposed in [6] and are explicitly shown within the tables. They were indispensable in solving the main problem of [6], namely the decomposition of products of fundamental representations of E_8.

7. Decomposition of symmetrized powers of orbits

7.1. Symmetrized powers of G-polynomials

The product of m identical polynomials, say $P(\lambda; x)$, is the subject of the action of the permutation group S_m of m elements. Thus it can be decomposed into a sum of components with a specific permutation symmetry. It is well known from representation theory that the permutation symmetry commutes with the action of the Weyl group. Consequently, each permutation symmetry component can be decomposed into a sum of polynomials.

Let $\square$ be short-hand notation for a polynomial (8). The product of two and more copies of $\square$ decomposes into the symmetry components indicated by their Young tableaux:

$$
\square\otimes\square = \begin{array}{c}\square\\ \square\end{array} + \square\square, \qquad \square\otimes\square\otimes\square = \begin{array}{c}\square\\ \square\\ \square\end{array} + 2\begin{array}{ll}\square & \square\\ \square & \end{array} + \square\square\square, \dots . \tag{13}
$$

In general, the square stands for a set of G-invariant items, each square containing the same items. Those can be monomials of a polynomial, or weights in the case of the weight system of a representation of a semisimple Lie group/algebra, or points of a G-orbit. The product of m copies of the same square decomposes into permutational symmetry components according to the representations of the group S_m. The components are identified by their Young tableau. Each of the components is further decomposable into the sum of parts that are labeled by the orbits of the Coxeter group G.

In order to perform such a two-step decomposition, (i) the items of the square need to be numbered consecutively in any convenient way. The items belonging to a particular permutation symmetry component are then determined according to the inequalities, shown in the next subsection, and more generally implied by the corresponding Young tableau. Then (ii) items belonging to a particular Young tableau, which are labeled by points transformed by G, are sorted out into the Coxeter group orbits. Practically it suffices to find the items labeled by dominant points.

7.2. Symmetrized powers of G-orbits

For simplicity of notation let us continue to label an orbit $G(\lambda)$ by its dominant point λ. The product of the same two G-orbits decomposes into its symmetric and antisymmetric parts:

$$\lambda \otimes \lambda = (\lambda^2)_{\text{symm}} \cup (\lambda^2)_{\text{anti}}. \tag{14}$$

Each of the two terms on the right-hand side is further decomposable into the sum of individual orbits. Let $\lambda_1, \lambda_2, \ldots$ be the points of the orbit λ numbered in any order. Then the content of the two parts is determined by the following inequalities:

$$(\lambda^2)_{\text{symm}} \ni \lambda_p + \lambda_q, \qquad p \geqslant q, \tag{15}$$

$$(\lambda^2)_{\text{anti}} \ni \lambda_p + \lambda_q, \qquad p > q. \tag{16}$$

The product of three copies of λ decomposes likewise

$$\lambda \otimes \lambda \otimes \lambda = (\lambda^3)_{\text{symm}} \cup (\lambda^3)_{\text{anti}} \cup 2(\lambda^3)_{\text{mixed}}, \tag{17}$$

where permutation symmetry components are formed from the N points as follows:

$$(\lambda^3)_{\text{symm}} \ni \lambda_p + \lambda_q + \lambda_s, \qquad p \geqslant q \geqslant s, \tag{18}$$

$$(\lambda^3)_{\text{anti}} \ni \lambda_p + \lambda_q + \lambda_s, \qquad p > q > s, \tag{19}$$

$$(\lambda^3)_{\text{mixed}} \ni \lambda_p + \lambda_q + \lambda_s, \qquad p \geqslant q \quad \text{and} \quad p > s. \tag{20}$$

Similarly, any higher power decomposes into permutation symmetry components where each is a sum of individual orbits.

7.3. Two-dimensional examples

$$\begin{aligned} A_2: \quad & (0,1)^2_{\text{symm}} = (1,0) \cup (0,2), \\ & (0,1)^2_{\text{anti}} \ = (1,0). \\ & (1,1)^2_{\text{symm}} = (2,2) \cup (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0), \\ & (1,1)^2_{\text{anti}} \ = (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0). \\ & (1,0)^3_{\text{symm}} = (1,1) \cup (3,0) \cup (0,0), \end{aligned}$$

$$(1,0)^3_{\text{anti}} = (0,0),$$
$$(1,0)^3_{\text{mixed}} = (1,1) \cup 2(0,0).$$

$$C_2: \quad (0,1)^2_{\text{symm}} = (2,0) \cup (0,2) \cup 2(0,0),$$
$$(0,1)^2_{\text{anti}} = (2,0) \cup 2(0,0).$$
$$(1,0)^3_{\text{symm}} = (1,1) \cup (3,0) \cup 2(1,0),$$
$$(1,0)^3_{\text{anti}} = (1,0),$$
$$(1,0)^3_{\text{mixed}} = (1,1) \cup 3(1,0).$$

$$G_2: \quad (0,1)^2_{\text{symm}} = (1,0) \cup (0,2) \cup (0,1) \cup 3(0,0),$$
$$(0,1)^2_{\text{anti}} = (1,0) \cup (0,1) \cup 3(0,0).$$
$$(1,0)^3_{\text{symm}} = (1,3) \cup (3,0) \cup (2,0) \cup 3(1,0) \cup 2(0,3) \cup 2(0,0),$$
$$(1,0)^3_{\text{anti}} = (2,0) \cup 2(1,0) \cup 2(0,0),$$
$$(1,0)^3_{\text{mixed}} = (1,3) \cup 2(2,0) \cup 5(1,0) \cup 2(0,3) \cup 4(0,0).$$

$$H_2: \quad (0,1)^2_{\text{symm}} = (\tau,0) \cup (0,2) \cup (0,\tau-1),$$
$$(0,1)^2_{\text{anti}} = (\tau,0) \cup (0,\tau-1).$$
$$(1,0)^3_{\text{symm}} = (2-\tau,1) \cup (1,\tau) \cup (3,0) \cup (\tau,0) \cup (0,\tau-1),$$
$$(1,0)^3_{\text{anti}} = (\tau,0) \cup (0,\tau-1),$$
$$(1,0)^3_{\text{mixed}} = (2-\tau,1) \cup (1,\tau) \cup 2(\tau,0) \cup 2(0,\tau-1).$$

7.4. Three-dimensional examples

$$A_3: \quad (1,0,0)^3_{\text{symm}} = (1,1,0) \cup (3,0,0) \cup (0,0,1),$$
$$(1,0,0)^3_{\text{anti}} = (0,0,1),$$
$$(1,0,0)^3_{\text{mixed}} = (1,1,0) \cup 2(0,0,1).$$

$$B_3: \quad (1,0,0)^3_{\text{symm}} = (1,1,0) \cup (3,0,0) \cup 3(1,0,0) \cup (0,0,2),$$
$$(1,0,0)^3_{\text{anti}} = 2(1,0,0) \cup (0,0,2),$$
$$(1,0,0)^3_{\text{mixed}} = (1,1,0) \cup 5(1,0,0) \cup 2(0,0,2).$$

$$C_3: \quad (1,0,0)^2_{\text{symm}} = (2,0,0) \cup (0,1,0) \cup 3(0,0,0),$$
$$(1,0,0)^2_{\text{anti}} = (0,1,0) \cup 3(0,0,0).$$

$$H_3: \quad (1,0,0)^2_{\text{symm}} = (2,0,0) \cup (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0),$$
$$(1,0,0)^2_{\text{anti}} = (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0).$$

8. Congruence classes, indices and anomaly numbers of polytopes

Here we introduce numerical characterizations of W-orbits, analogs of similar quantities known for irreducible representations of semisimple Lie groups, which proved particularly useful in their application.

8.1. Congruence classes

Inclusion among the lattices (4) is an important property of the Weyl group W. The weight lattice P can be understood as a union of several components, each isomorphic to the root lattice Q. The components are shifted relative to each other by some elements of P. An individual component consists of points belonging to one congruence class of P. The index of Q in P, denoted $|Z|$, is the number of distinct congruence classes in P. The value of $|Z|$ reflects other properties of G. For example, it is the order of the center of G, it is a common denominator of coordinates of all points of P when given in the basis of simple roots, etc. One has $|Z| > 1$ for all G but for the exceptional simple Lie groups of types E_8, F_4 and G_2.

The congruence number c is a number attached to points of P. The value of c is common to all points of the same congruence class. It can be defined in a number of equivalent ways. Our definition coincides with that of [33]. All points of any W-orbit belong to the same congruence class. Furthermore, orbits obtained from the decomposition of a product belong to the same congruence class, and their congruence number is the sum of the congruence numbers of the orbits of the multiplication. That is also true for the decomposition of symmetrized powers of orbits.

Let $x = (x_1, x_2, \ldots, x_n) \in P$ be a point to consider in the ω-basis. Its congruence number $c(x)$ is given by the following formulae:

$$\begin{aligned}
A_n : &\quad c(x) = \sum_{k=1}^{n} k x_k \quad \mod (n+1)\\
B_n : &\quad c(x) = x_n \quad \mod 2\\
C_n : &\quad c(x) = \sum_{k=1}^{[\frac{n+1}{2}]} x_{2k-1} \quad \mod 2\\
D_n : &\quad c(x) = (c_1(x) \quad \mod 2, c_2(x) \quad \mod 4),\\
&\quad c_1(x) = x_{n-1} + x_n\\
&\quad c_2(x) = \begin{cases} 2x_1 + 2x_3 + \cdots + 2x_{n-2} + (n-2)x_{n-1} + n x_n, & n \text{ odd}\\ 2x_1 + 2x_3 + \cdots + 2x_{n-3} + (n-2)x_{n-1} + n x_n, & n \text{ even}\end{cases}\\
E_6 : &\quad c(x) = x_1 - x_2 + x_4 - x_5 \quad \mod 3\\
E_7 : &\quad c(x) = x_4 + x_6 + x_7 \quad \mod 2.
\end{aligned} \tag{21}$$

For E_8, F_4 and G_2 there is only one congruence class, namely $c(x) = 0$ for all $x \in P$. Note also that the roots of any group belong to the congruence class $c(x) = 0$. Hence also the points of the root lattice of any group belong to the congruence class $c(x) = 0$.

The points of any single G-orbit belong to the same congruence class because the difference between any two points of the same orbit is an integer linear combination of simple roots, as can be derived from (6).

For the non-crystallographic groups, the congruence classes can be similarly defined, involving their appropriate irrationality. It is important to recall that, in these cases, P is a dense lattice. The coordinates of $x \in P$, relative to the ω-basis, are the numbers $a + \tau b$, with $a, b \in \mathbb{Z}$:

$$H_2 \quad : \quad c(x) = \tau x_1 + 2x_2 \quad \mod 5, \qquad \text{where} \quad \tau = 3. \tag{22}$$

8.2. The second and higher indices

The second and higher indices were defined [34] for weight systems of irreducible finite-dimensional representations of compact semisimple Lie groups. Extensive tables of indices of degree 0, 2 and 4 are found in [26]. The fact that a weight system is a union of several W-orbits suggests that the indices could be introduced for individual orbits. Moreover, we introduce them also for non-crystallographic Coxeter groups with the same formulae.

For any finite Coxeter group G, we define an index $I_\lambda^{(2k)}$ of degree $2k$ of a G-orbit $G(\lambda)$ by

$$I_\lambda^{(2k)} = \sum_{\mu\in G(\lambda)} \langle\mu,\mu\rangle^k = \langle\lambda,\lambda\rangle^k I_\lambda^{(0)}, \qquad k = 0, 1, 2, \ldots, \tag{23}$$

because points of $G(\lambda)$ are equidistant from the origin. Clearly $I_\lambda^{(0)} = |G(\lambda)|$ is the number of points of the orbit $G(\lambda)$ given by (5).

Higher indices of products of two orbits, $G(\lambda_1) \otimes G(\lambda_2)$, are also useful in calculating the decompositions. Let r be the rank of G:

$$I^{(2k)}(G(\lambda_1)\otimes G(\lambda_2)) = I^{(2k)}_{\lambda_1\otimes\lambda_2} = I^{(2k)}_{\lambda_1+\lambda_2} + \cdots + I^{(2k)}_{\lambda_1+\overline{\lambda_2}} \tag{24}$$

$$I^{(0)}_{\lambda_1\otimes\lambda_2} = I^{(0)}_{\lambda_1} I^{(0)}_{\lambda_2} \tag{25}$$

$$I^{(2)}_{\lambda_1\otimes\lambda_2} = I^{(2)}_{\lambda_1} I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(2)}_{\lambda_2} \tag{26}$$

$$= I^{(0)}_{\lambda_1} I^{(0)}_{\lambda_2} (\langle\lambda_1,\lambda_1\rangle + \langle\lambda_2,\lambda_2\rangle) \tag{27}$$

$$I^{(4)}_{\lambda_1\otimes\lambda_2} = I^{(4)}_{\lambda_1} I^{(0)}_{\lambda_2} + \frac{2(r+2)}{r} I^{(2)}_{\lambda_1} I^{(2)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(4)}_{\lambda_2}. \tag{28}$$

Table 4 presents examples of indices of degree 0, 2, 4, 6 and 8 of individual orbits of A_2, C_2, G_2 and H_2.

8.3. Anomaly numbers

Triangle anomaly numbers were introduced in physics [27, 35, 36] as quantities assigned to irreducible representations of a few compact semisimple Lie groups and calculated from the weight systems of their representations. Constraints on possible models in particle physics were imposed in terms of admissible values of the anomaly numbers of representations involved in a particular model. Generalization of the concept to all compact semisimple Lie groups and to higher than third degree anomaly number originates in [37]. Our goal here is to show that the anomaly numbers can be used also for constituents of the weight systems of irreducible representations, namely for W-orbits and more generally, for the orbits $G(\lambda)$ of any finite Coxeter group.

The anomaly number $I_\lambda^{(2k-1)}$ of degree $2k-1$ of the orbit $G(\lambda)$ of the Coxeter group G is defined as follows:

$$I_\lambda^{(2k-1)} = \sum_{\mu\in G(\lambda)} \langle\mu,u\rangle^{2k-1}, \qquad k = 1, 2, \ldots, \tag{29}$$

where u is a special vector passing through the origin. In particular, $I^{(1)} = 0$ in all cases. The anomaly number of physics literature is $I^{(3)}$, therefore it is the only one we consider.

Frequently used property of $I^{(3)}$ is the decomposition of the product of two orbits, which is the analog of (26):

$$I^{(3)}_{\lambda_1\otimes\lambda_2} = I^{(3)}_{\lambda_1} I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1} I^{(3)}_{\lambda_2} = I^{(3)}_{\lambda_1+\lambda_2} + \cdots + I^{(3)}_{\lambda_1+\overline{\lambda_2}}. \tag{30}$$

In general terms, the direction of u can be characterized as follows. Suppose W in (29) is the Weyl group of a compact simple Lie group G, and that G has a maximal reductive subgroup

Table 4. Examples of the indices $I^{(2k)}$, $k = 0, \ldots, 4$.

A_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
(1, 0)	3	2	4	8	16
(2, 0)	3	8	64	512	4096
(1, 1)	6	12	72	432	2592
(2, 1)	6	28	392	5488	76832
C_2	$I^{(0)}$	$I^{(2)}$	$2I^{(4)}$	$4I^{(6)}$	$8I^{(8)}$
(1, 0)	4	2	2	2	2
(0, 1)	4	4	8	16	32
(2, 0)	4	8	32	128	512
(0, 2)	4	16	128	1024	8192
(1, 1)	8	20	100	500	2500
(2, 1)	8	40	400	4000	40000
G_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
(0, 1)	6	4	8	16	32
(1, 0)	6	12	72	432	2592
(0, 2)	6	16	128	1024	8192
(0, 3)	6	36	648	11664	209952
(2, 0)	6	48	1152	27648	663552
(1, 1)	12	56	784	10976	153664
H_2	$I^{(0)}$	$(3-\tau)I^{(2)}$	$(3-\tau)^2 I^{(4)}$		
(1, 0)	5	10	20		
(2, 0)	5	40	320		
(1, 1)	10	$20(\tau+2)$	$40(\tau+2)^2$		
(2, 1)	10	$10(4\tau+10)$	$10(4\tau+10)^2$		

of type $U(1) \times G'$. Then the direction of u is given by the direction corresponding to $U(1)$ in the Euclidean space spanned by the roots of G.

The first question to answer is when such a maximal subgroup is present. For a complete list of the cases see below [38]:

$$\begin{array}{lll}
A_n \supset A_{n-1} \times U(1) & n \geqslant 2 & \\
A_n \supset A_k \times A_{n-k-1} \times U(1) & n \geqslant 3, & 1 \leqslant k \leqslant \left[\dfrac{n-1}{2}\right] \\
B_n \supset B_{n-1} \times U(1) & n \geqslant 3 & \\
C_n \supset A_{n-1} \times U(1) & n \geqslant 2 & \\
D_n \supset A_{n-1} \times U(1) & n \geqslant 4 & \\
D_n \supset D_{n-1} \times U(1) & n \geqslant 5 & \\
E_6 \supset D_5 \times U(1) & & \\
E_7 \supset E_6 \times U(1). & &
\end{array} \tag{31}$$

As long as each orbit of a given group contains with every weight also its negative, the anomaly numbers are equal to zero. Therefore the interesting cases that remain are found in A_n, D_{2k+1}, E_6 and E_7. In physics, however, only the anomaly numbers of $A_n \supset A_{n-1} \times U(1)$ are used so far.

Anomaly numbers of H_2, H_3 and H_4 are also defined by (29). In those cases, however, the direction of u has to be determined differently since there is no $U(1)$ subgroup. Instead, one can require that u be orthogonal to selected simple roots: α_1 for H_2, α_1 and α_2 for H_3, and α_1, α_2, and α_3 for H_4. Anomaly numbers for H_2 are zero for all orbits. They will be considered elsewhere [28], along with the anomaly numbers of other non-crystallographic groups.

9. Concluding remarks

(1) Useful and interesting objects may turn out to be G-orbits with each point decorated by a sign [19] according to the following rule. The dominant point, say λ, and all points obtained from it by an even number of reflections generating G carry a positive sign, while all points of the orbit obtained from λ by an odd number of reflections carry a negative sign. Let us call an S-orbit a decorated orbit of λ of G, while the orbits without the sign decoration, i.e. all positive signs, are called C-orbits of λ of G. In order to avoid ambiguities, it should be stipulated that λ of an S-orbit must have all coordinates positive in ω-basis.

Multiplication of such orbits follows simple rules:

$$C\text{-orbit} \times C\text{-orbit} \longrightarrow C\text{-orbits}, \tag{32}$$

$$C\text{-orbit} \times S\text{-orbit} \longrightarrow S\text{-orbits}, \tag{33}$$

$$S\text{-orbit} \times S\text{-orbit} \longrightarrow C\text{-orbits}. \tag{34}$$

In (32), all coefficients in the decomposition of the product are positive integers, while in (33) and (34) all such coefficients are integers, but not all may be positive.

The decomposition of many products of C-orbits with lowest nontrivial S-orbit can be directly inferred from the tables [25], using the Weyl character formula.

(2) In the examples, we often required that a G-orbit consist of points of the weight lattice P. Very few properties of the orbits would have been lost, had we instead allowed $\lambda \in \mathbb{R}^n$. The congruence classes would not then be applicable.

Consider the following products of A_2 orbits as examples:

$$(a, 0) \otimes (\varepsilon, 0) = (a + \varepsilon, 0) \cup (a - \varepsilon, \varepsilon), \qquad 0 < \varepsilon \ll 1, \quad a \geqslant 1, \tag{35}$$

$$(a, 0) \otimes (0, a + \varepsilon) = (a, a + \varepsilon) \cup (0, \varepsilon), \qquad 0 < \varepsilon \ll 1, \quad a \gg 1. \tag{36}$$

The radii of the two orbits in the decomposition (35) can be drawn arbitrarily close by a suitable choice of ε, and in (36) they can be pushed as far apart as desired by the choice of a. The second orbit in (36) has a radius equal to $\varepsilon\sqrt{\frac{2}{3}}$.

(3) For a geometric interpretation of orbits as polytopes, refer to the paragraph following equation (12). The 'interaction' (i.e. product) between two concentric orbit layers results in the layered structure of orbits. They are subject to the equality of indices of various degrees, congruence numbers, relations between anomaly numbers. Speculative interpretation can go further: consider $I^{(2)}_{\lambda}$ as the 'energy' of the orbit and $I^{(2)}_{\lambda\otimes\lambda'}$ as the 'energy' of the interacting pair, etc.

(4) Although we did not pursue it here, orbit multiplication can be viewed as an 'interaction' between two orbits similarly as used in particle physics to view interacting multiplets of particles. A multiplet is described by the weight system of an irreducible representation of the corresponding Lie group/algebra. Here, the role of the multiplet would be given to the set of points of an orbit. In both cases, such interactions would be governed by the strict equality of indices of various degrees, congruence numbers, relations between

anomaly numbers. But there is a price to pay for such a reinterpretation of multiplets: the overall invariance of the theory with respect to the Lie group would be reduced to the invariance with respect to the Coxeter group or to its (discrete) image 'lifted' into the Lie group [39].

(5) It would be useful to ask additional questions about the properties of indices and anomaly numbers of various degrees. Such questions can be answered by adaptation of the methods used for the weight system of representations [34, 37].

(6) In place of finite Coxeter groups, we could have chosen to consider other finite groups for similar considerations [40]. The immediate motivations for our choice were recent applications in harmonic analysis, where W-orbits play a fundamental role. Equally interesting would be to consider orbits of infinite Coxeter groups. (A Coxeter group with connected diagram is of infinite order if its diagram is different from those listed in section 2.) The orbits of representations of Kac–Moody algebras would be relatively easily amenable to such a study.

(7) Similarly, we could consider orbits of two or more seed points. A simple example is the root system of the group G_2. Choosing as the two seed points one short root and one long root, say α_2 and $\alpha_1 + 3\alpha_2$, the orbit of the pair is a star-like polygon formed by the root system of G_2.

(8) An interesting problem appears to be to pursue a similar study of orbits of the even subgroups of Coxeter groups, particularly because these subgroups are not Coxeter groups in general.

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